On Periodic 3D Navier-Stokes Equations when the initial velocity is in L^2 and the initial vorticity is in L^1.

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Abstract

This paper is devoted to the 3D Navier-Stokes equations in a periodic case. Assuming that the initial data $\mathbf{u}_0$ is in L^2_x while the initial vorticity $\omega_0 = \nabla \times \mathbf{u}_0$ is in L^1_x, we prove the existence of a distributional solution $(\mathbf{u}, p)$ to the Navier-Stokes equations such that $\mathbf{u} \in L^2_t(H^1_x) \cap L^\infty_t(L^2_x) \cap L^p_t(W^{2,p}_x)$ $\forall p < 5/4$, and $\omega = \nabla \times \mathbf{u} \in L^\infty_t(L^1_x) \cap L^p_t(W^{1,p}_x)$ $\forall p < 5/4$, $p \in L^{5/4}_t(W^{1,5/4}_x)$. The main remark of the paper is that the equation for the vorticity can be considered as a parabolic equation with a right hand side in $L^1_{t,x}$. Thus one can use tools of the renormalization theory. Studying approximations deduced from a Large Eddy Simulations model, we focus our attention in passing to the limit in the equation for the vorticity. Finally, we look for sufficient conditions yielding uniqueness of the limit.

1 Introduction and main results

We consider the Navier-Stokes equations, posed in a Q-periodic case in $I\!R^3$ ($Q = [0, 2\pi]^3$),

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\nabla)\,\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = 0, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u}_{t=0} = \mathbf{u}_0, \end{cases} \tag{1.1}$$

subject to the constrain that the unknown field $(\mathbf{u}, p)$ has zero mean on Q.

Let

$$V = \left\{ \mathbf{v} \in L^2_x,\ \nabla \cdot \mathbf{v} = 0 \right\}, \tag{1.2}$$

where L^2_x is the space of all the Q-periodic functions (vector fields or tensors) with restriction to Q in $L^2(Q)$ (with their components restricted to Q in $L^2(Q)$) and such that their mean value on Q are equal to zero, that is

$$\int_Q \mathbf{v} = 0.$$

In the context of regularity, we treat functions, vector fields and any tensor with the same formalism for the sake of simplicity. Generally speaking, if $E(X)$ is a generic space function on X, E_x denotes all the Q-periodic functions (vector fields or tensors) with their restriction on Q in $E(Q)$ and with zero mean value on Q.

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Throughout the paper, we assume that

$$\mathbf{u}_0 \in V. \tag{1.3}$$

We know since the work of J. Leray [15] that equation (1.1) has a distributional solution $\mathbf{u} \in L^2_t(H^1_x) \cap L^\infty_t(L^2_x)$, time continuous in L^2_x for its weak topology. J. Leray was considering approximations given by (1.4),

$$\begin{cases} \partial_t \mathbf{w} + ((\mathbf{w} \star \rho_\delta)\nabla)\mathbf{w} - \nu\Delta\mathbf{w} + \nabla q = 0, \\ \nabla \cdot \mathbf{w} = 0, \\ \mathbf{w}_{t=0} = \mathbf{u}_0 \star \rho_\delta, \end{cases} \tag{1.4}$$

where ρ_δ is a molifier. The solution that he constructed was obtained as limit of subsequences of solution of (1.4). J. Leray called such a solution to the Navier-Stokes equations a "turbulent solution". We shall call them "Leray's solution". We do not know if there is a unique Leray's solution.

We consider in this paper the following approximations to the Navier-Stokes equations:

$$\begin{cases} \partial_t \mathbf{w} + \nabla \cdot (A_\delta^{-1}(\mathbf{w}\mathbf{w})) - \nu\Delta\mathbf{w} + \nabla q = 0, \\ \nabla \cdot \mathbf{w} = 0, \\ \mathbf{w}_{t=0} = A_\delta^{-1}\mathbf{u}_0, \end{cases} \tag{1.5}$$

where $A_\delta\phi := -\delta^2\Delta\phi + \phi$, $(\mathbf{w}\mathbf{w})_{ij} = w^i w^j$ for $\mathbf{w} = (w^1, w^2, w^3)$; A_δ^{-1} is called the LES filter.

Approximations (1.5) have been introduced in [13] in the context of Large Eddy Simulations modelling for turbulent flows (LES). For a general overview of LES, the reader is refered to the book of P. Sagaut [23] or the review of W. Layton [11]. The goal is to filter small eddies of size less δ in a turbulent flow. In [13], we were looking for a simple tractable model yielding smooth approximations and which ovoids eddy viscosities. In [12] we have shown that (1.5) has a unique solution with a reasonnable regularity (see Theorem 2.1 below).

The introduction of (1.5) has been motivated by the fact that LES models with eddy viscosities give rise to very weak solutions which cannot be proved having any regularity properties (see in [14]). Moreover, model (1.5) presents the advantage that it allows to create programs with very simple numerical algorithms. Numerical simulations are in progress.

The main motivation of this paper is the mathematical study of the behavior of sequences of approximations constructed by (1.5) to validate the numerical schemes using them for simulate turbulent flows. We have shown in [13] that up to a subsequence, approximations due to (1.5) converge when δ goes to zero toward a distributional solution to the Navier-Stokes equations. We call such a solution a LES solution. We do not know if this LES solution is unique. We look for more convergence properties of the LES sequences in what follows.

We first prove that the classical energy inequality is satisfied by each LES solution (see (1.17) below), a result not yet proved in the previous papers. Next, we consider the case where the initial vorticity $\omega_0 = \nabla \times \mathbf{u}_0$ is in L^1_x. In the 3D case under our scope, the equation for the vorticity is

$$\begin{cases} \partial_t \omega + (\mathbf{u}\nabla)\omega - \nu\Delta\omega = (\omega\nabla)\mathbf{u}, \\ \omega_{t=0} = \omega_0 = \nabla \times \mathbf{u}_0. \end{cases} \tag{1.6}$$

The reader can found a derivation of this equation in the book of G. -K. Batchelor [3], equation 5.2.2 page 267.

Energy inequality tells that $\mathbf{u} \in L^2_t(H^1_x)$. Therefore the source term $(\omega\nabla)\mathbf{u}$ is naturally in $L^1_{t,x}$. Thus, (1.6) can be treated as a parabolic equation with a second hand side in $L^1_{t,x}$. This allow an approach using some elements of the renormalization theory, never used before in the context of the Navier-Stokes equations as far as we know.

The notion of renormalized solution is originally due to P. -L. Lions and R. -J. Di Perna for Kinetic models (see for instance [8] and more references inside). This notion has been improved by F. Murat [22] in the case of elliptic equations and by D. Blanchard and F. Murat in the parabolic case [4].

The main difficulty consists in proving estimates by a rigorous analysis. Yet these estimates are quickly deduced from formal considerations. Starting from a LES solution $(\mathbf{u}, p)$, we work with the approximations deduced from our LES filter A_δ^{-1}. One notes $\mathbf{w}_n$ the LES approximations for the velocity (*i.e* the unique solution to (1.5) for δ_n where δ_n goes to zero) and $\omega_n = \nabla \times \mathbf{w}_n$, solution of

$$\left\{ \begin{array}{l} \partial_t \omega_n + A_{\delta_n}^{-1}((\mathbf{w}_n\nabla)\omega_n) - \nu\Delta\omega_n = A_{\delta_n}^{-1}(((\nabla \times \mathbf{w}_n)\nabla)\mathbf{w}_n), \\ (\omega_n)_{t=0} = A_{\delta_n}^{-1}(\omega_0) = \nabla \times (A_{\delta_n}^{-1}\mathbf{u}_0). \end{array} \right. \tag{1.7}$$

We show in the paper that the sequence $(\omega_n)_{n\in\mathbb{N}}$ converges in suitable spaces up to a subsequence to $\omega = \nabla \times \mathbf{u}$, distributional solution to (1.8),

$$\left\{ \begin{array}{l} \partial_t \omega + (\mathbf{u}\nabla)\omega - \nu\Delta\omega = ((\nabla \times \mathbf{u})\mathbf{u}, \\ \omega_{t=0} = \omega_0 = \nabla \times \mathbf{u}_0 \in L^1_x. \end{array} \right. \tag{1.8}$$

Hence ω is also solution to (1.6). The following regularity holds for ω:

$$\omega \in L^\infty_t(L^1_x), \tag{1.9}$$

$$\forall\, k \in \mathbb{R}^+, \quad T_k(\omega) \in L^2_t(H^1_x), \tag{1.10}$$

$$\omega \in \bigcap_{p<5/4} L^p_t(W^{1,p}), \tag{1.11}$$

where T_k denotes the truncation function at height k (see (5.12) below) and if $\omega = (\omega^1, \omega^2, \omega^3)$, $T_k(\omega)$ is the vector field $(T_k(\omega^1), T_k(\omega^2), T_k(\omega^3))$.

We know that a $L^\infty_t(L^1_x)$ bound for the vorticity has been first obtained by P. Constantin [7] and generalized By P. -L. Lions [20]. Estimate (1.9) can be also obtained when ω_0 is a bounded measure with the same techniques that we use. Regularity (1.10) and (1.11) seem to be a new contribution as far as we know. Notice that we have choosen to restrict ourselve to the case where ω_0 is in L^1_x unless the case where ω_0 is a bounded measure only for the sake of simplify the presentation.

The results of Blanchard-Murat [4] tell that equation (1.8) admits a unique renormalized solution $\tilde{\omega}$. We do not know if $\tilde{\omega} = \omega$. This is due to the lack of strong compactness in $L^1_{t,x}$ of the sequence $(((\nabla\times\mathbf{w}_n)\nabla\mathbf{w}_n))_{n\in\mathbb{N}}$ which can presents oscillations. We suppose that the notion of H-measures introduced by L. Tartar [25] can be adapted to study this question but we did not have explored yet this direction. Nevertheless, the Kolmogorov's laws say that frequencies larger than $O(R^{-3/4})$ in the flow are damped, where R is the Reynolds number (see for in instance in the book of U. Frisch [9]). Therefore, such oscillations do not arise physically.

The equation for the pressure can also be considered as an equation with a second hand side in L^1_x. Indeed, thanks to the incompressibilty, it easy seen that the pressure p is solution of

$$-\Delta p = -\nabla \mathbf{u} : \nabla \mathbf{u}^t \in L^1_{t,x}. \tag{1.12}$$

Formally speaking, tools linked to the renormalization should be usefull. For instance, one can wait that $T_k(p) \in L^1_t(H^1_x)$ for each $k > 0$. Unfortunatly, because of degeneracy in time of equation (1.12), very poor results are available. We are just able to prove that for g in a suitable class of bounded functions, $(g(q_n))_{n\in\mathbb{N}}$ is bounded in $L^2_t(H^1_x)$ where $(q_n)_{n\in\mathbb{N}}$ is the corresponding approximation for the pressure. This is mainly due to the fact that the sequence $(q_n)_{n\in\mathbb{N}}$ can developp very high oscillations at infinity and compactness properties are cruelly missing. Such oscillations are numerically observed in high resolution codes. Therefore we are constrain to use output filter to damp these numerical oscillations (see in [10]). Anyone who have tried in his life to write a simulation code for real turbulent flows knows that additional terms have to be had in the equations to stabilize the pressure.

The consequences of our study can be summerized in the following statement.

Theorem 1.1 *Let $\mathbf{u}_0 \in V$ such that $\nabla \times \mathbf{u}_0 = \omega_0 \in L^1_x$. Then the 3D periodic Navier-Stokes equations* (1.1) *has one distributional solution* $(\mathbf{u}, p)$ *with zero mean value, called LES solution, and such that:*

$$\mathbf{u} \in L^2_t(H^1_x) \cap L^\infty_t(L^2_x), \tag{1.13}$$

$$\mathbf{u} \in \bigcap_{p<5/4} L^p_t(W^{2,p}_x), \tag{1.14}$$

$$p \in L^{5/4}_t(W^{1,5/4}_x), \tag{1.15}$$

$$\partial_t u \in \bigcap_{p<5/4} L^p_{t,x}, \tag{1.16}$$

and which satisfies the energy inequality for each $t > 0$

$$\frac{1}{2}\int_Q |\mathbf{u}(t,x)|^2 dx + \nu \int_0^t \int_Q |\nabla \mathbf{u}(t',x)|^2 dx dt' \leq \frac{1}{2}\int_Q |\mathbf{u}_0(x)|^2 dx. \tag{1.17}$$

The vorticity $\omega = \nabla \times \mathbf{u}$ satisfies (1.9), (1.10) *and* (1.11).

The paper is organized as follows. We start by the recall of the main results of [12] and [13] and make clear the notion of LES solution. Then we prove that any LES solution satisfy the energy inequality (1.17). Next, we study the equation for the pressure, pointing out the difficulties. Afterwards, we consider the equation for the vorticity. The hypothesis that $\omega_0 \in L^1_x$ plays a role at this step. The consequences on the regularity of a LES solution are investigated. We conclude the paper by further remarks on the problem of uniqueness of the LES solution by proving the following statement.

Proposition 1.1 *Let $\mathbf{u}_0 \in V$. Assume that any LES solution* $(\mathbf{u}, p)$ *to* (1.1) *satisfies the following two hypothesis:*

(H1) *$u \in L^\infty_t(L^3_x)$ and there exists a LES sequence of approximations $(\mathbf{w}_n)_{n\in\mathbb{N}}$ corresponding to $\mathbf{u}$ such that $(\mathbf{w}_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty_t(L^3_x)$.*

(H2) *One have*

$$\lim_{k\to\infty} ||\mathbf{u} - T_k(\mathbf{u})||_{L^\infty_t(L^3_x)} = 0, \tag{1.18}$$

where $T_k(\mathbf{u}) = (T_k(\mathbf{u}^1), T_k(\mathbf{u}^2), T_k(\mathbf{u}^3))$.

Then there is a unique LES solution $(\mathbf{u}, p)$ *to the Navier-Stokes equations such that the the following properties are satisfied.*

- *The following regularity holds.*

$$\frac{D\mathbf{u}}{Dt} = \partial_t \mathbf{u} + (\mathbf{u}\nabla)\mathbf{u} \in L^2_x(W'), \tag{1.19}$$

  where $W = \{\mathbf{v} \in H^1_x;\ \nabla \cdot \mathbf{v} = 0\}$.

- *The energy equality is satisfied,*

$$\frac{1}{2}\int_Q |\mathbf{u}(t,x)|^2 dx + \nu \int_0^t \int_Q |\nabla \mathbf{u}(t',x)|^2 dx dt' = \frac{1}{2}\int_Q |\mathbf{u}_0(x)|^2 dx. \tag{1.20}$$

- *The sequence of LES velocities* $(\mathbf{w}_n)_{n\in I\!N}$ *converges strongly towards* $\mathbf{u}$ *in* $L^2([0,T], H^1_x)$ *for each* $T > 0$.

We mention that P. -L. Lions and N. Masmoudi [21] have proved that there is a unique solution to the Navier-Stokes equation in the space $C^0_t(L^3_x)$ by using a duality method. We have tried here to relax $C^0_t(L^3_x)$ into a $L^\infty_t(L^3_x)$ bound on the approximations to obtain uniqueness of the limit. For someone familiar with turbulence modelling, this is like a closure hypothesis. But $L^\infty_t(L^3_x)$ seems to be not enough to obtain uniqueness. This is why we have considered the additional hypothesis (1.18), which is a regularity assumption in time lighter than C^0. Our proof uses mainly the regularity result (1.19) and the strong convergence of the approximations.

2 LES Approximations

2.1 Definition of an LES filter

The mean operator A_δ is defined by

$$A_\delta \overline{\phi} := -\delta^2 \Delta \overline{\phi} + \overline{\phi} = \phi, \tag{2.1}$$

with periodic conditions, and fields with mean value equal to zero. This defines an operator

$$A_\delta : W^{1,p}_x \to W^{-1,p}_x.$$

One easily sees that A_δ is self-adjoint and has the regularity property

$$\forall\, r, \quad \forall\, \phi \in H^r_x, \quad \overline{\phi} = A_\delta^{-1}\phi \in H^{r+2}_x. \tag{2.2}$$

We shall call he operator A_δ^{-1} the LES filter. Note that A_δ^{-1} commutes with every differential operator thanks to the periodic conditions. That means for instance

$$\partial_i A_\delta^{-1} \phi = A_\delta^{-1} \partial_i \phi \tag{2.3}$$

Remark 2.1 *For obvious reasons, all the fields that we consider are Q-periodic in space and have a zero mean value on Q equal to zero. We shall not mention it in the remainder.*

2.2 Approximations : existence and uniqueness

One consider the following problem, where $(\mathbf{w}, q)$ is the unknown Q-periodic field:

$$
\begin{cases}
\partial_t \mathbf{w} + \nabla \cdot (A_\delta^{-1}(\mathbf{w}\mathbf{w})) - \nu \Delta \mathbf{w} + \nabla q = 0, \\
\nabla \cdot \mathbf{w} = 0, \\
\mathbf{w}_{t=0} = A_\delta^{-1} \mathbf{u}_0, \\
\displaystyle\int_Q \mathbf{w} = 0, \quad \int_Q q = 0.
\end{cases}
\tag{2.4}
$$

Here we note $\mathbf{w}\mathbf{w} = \mathbf{w} \otimes \mathbf{w} = (\mathbf{w}_i \mathbf{w}_j)_{1 \le i,j \le 3}$ and we recall that when $\nabla \cdot \mathbf{w} = 0$, one has $\nabla \cdot (\mathbf{w}\mathbf{w}) = (\mathbf{w}\nabla)\mathbf{w}$.

Results of [12] and [13] can be summerized as follows.

Theorem 2.1 *Assume that $\mathbf{u}_0 \in V$. Then the model* (2.4) *has a unique weak solution*

$$
(\mathbf{w}_\delta, q_\delta) \in [L^2_t(H^2_x) \cap L^\infty_t(H^1_x)] \times L^2_t(L^2_x)
\tag{2.5}
$$

and the energy equality holds for all $t > 0$:

$$
\begin{cases}
\displaystyle\frac{1}{2} \int_Q (|\mathbf{w}_\delta(t,x)|^2 dx + \delta^2 \int_Q |\nabla \mathbf{w}_\delta(t,x)|^2)\, dx + \\
\qquad\qquad \displaystyle\nu \int_0^t \int_Q (|\nabla \mathbf{w}_\delta(t',x)|^2 + \delta^2 |\Delta \mathbf{w}_\delta(t',x)|^2)\, dx dt' = \\
\qquad\qquad\qquad\qquad \displaystyle\frac{1}{2} \int_Q (|\overline{\mathbf{u}_0}(x)|^2 + \delta^2 |\nabla \overline{\mathbf{u}_0}(x)|^2)\, dx,
\end{cases}
\tag{2.6}
$$

where $\overline{\mathbf{u}_0} = A_\delta^{-1}\mathbf{u}_0$. In addition if $\mathbf{u}_0 \in V \cap H^{k-1}$ $(k \ge 1)$, then

$$
(\mathbf{w}_\delta, q_\delta) \in [L^2_t(H^{k+2}_x) \cap L^\infty_t(H^{k+1}_x)] \times L^2_t(H^k_x).
\tag{2.7}
$$

2.3 Convergence towards Navier-Stokes equations

Recall one result proved in [12].

Theorem 2.2 *There is a sequence $\delta_n \to 0$ as $n \to \infty$ such that*

$$
(\mathbf{w}_{\delta_n}, q_{\delta_n}) \to (\mathbf{u}, p) \quad as \quad n \to \infty
$$

where

$$
(\mathbf{u}, p) \in [L^\infty_t(L^2_x) \cap L^2_t(H^1_x)] \times L^{\frac{4}{3}}_t(L^2_x)
\tag{2.8}
$$

is a solution of the Navier-Stokes equations (1.1) *in the sense of the distributions. The sequence $(w_{\delta_n})_{n \in \mathbb{N}}$ converges strongly to $\mathbf{u}$ in $L^2_t(L^2_x)$,* a.e *and weakly in $L^2_t(H^1_x)$ while the sequence $(q_{\delta_n})_{n \in \mathbb{N}}$ converges weakly to p in the space $L^{\frac{4}{3}}_t(L^2_x)$.*

Definition 2.1 *We shall say that $\mathbf{u} = \mathbf{u}(t,x)$ is in the set $\mathcal{V}(\mathbf{u}_0)$ and $p = p(t,x)$ in the set $\mathcal{P}(\mathbf{u}_0)$ if and only if*

$$
(\mathbf{u}, p) \in [L^\infty_t(L^2_x) \cap L^2_t(H^1_x)] \times L^{\frac{4}{3}}_t(L^2_x)
$$

is a distributional solution to the Navier-Stokes equations (1.1) *and is a limit of a subsequence of the sequence $(\mathbf{w}_\delta, q_\delta)$ solution of* (2.4)*. We shall say that $(\mathbf{u}, p)$ is a LES solution to the Navier-Stokes equations.*

Notice that Theorem 2.2 makes sure that $\mathcal{V}(\mathbf{u}_0) \neq \emptyset$, $\mathcal{P}(\mathbf{u}_0) \neq \emptyset$.

3 Classical estimates

3.1 Energy estimates

In the following one writes $\mathbf{u} = (u_1, u_2, u_3)$ and uses the convention of repeated index sommation. For a fixed $\mathbf{u}_0 \in V$, one puts

$$||\mathbf{u}_0||^2_{L^2_x} = E_0. \tag{3.1}$$

Given $\mathbf{u} = \mathbf{u}(t, x) \in \mathcal{V}(\mathbf{u}_0)$, one notes

$$E(t, \mathbf{u}) = \frac{1}{2}\int_Q |\mathbf{u}(t, x)|^2 dx + \nu \int_0^t \int_Q |\nabla \mathbf{u}(t', x)|^2 dx dt'. \tag{3.2}$$

Theorem 3.1 *Let $\mathbf{u}_0 \in V$, $\mathbf{u} \in \mathcal{V}(\mathbf{u}_0)$. Then $t \to E(t, \mathbf{u})$ is a non increasing function of t and one has in particular*

$$\forall t \in I\!R, \quad E(t, \mathbf{u}) \le E_0. \tag{3.3}$$

Corollary 3.1 *The following inequalities hold:*

$$||\mathbf{u}||_{L^\infty_t(L^2_x)} \le \sqrt{E_0}, \tag{3.4}$$

$$||\mathbf{u}||_{L^2_t(H^1_x)} \le \sqrt{\frac{E_0}{2\nu}}. \tag{3.5}$$

Proof of Theorem 3.1. The field $\mathbf{u}$ is limit of a subsequence of the sequence $(\mathbf{w}_\delta)_{\delta>0}$, subsequence still denoted by $(\mathbf{w}_\delta)_{\delta>0}$. One starts from the energy balance (2.6). Let

$$D_\delta(t) = \delta^2 \left(\int_Q |\nabla \mathbf{w}_\delta(t, x)|^2 dx + \nu \int_0^t \int_Q |\Delta \mathbf{w}_\delta(t', x)|^2 dx dt' \right).$$

Integrating (2.6) with respect to the time on the interval $[0, \tau]$ yields

$$\left\{ \begin{array}{l} \displaystyle \frac{1}{2}\int_0^\tau \int_Q |\mathbf{w}_\delta(t, x)|^2 dx dt + \nu \int_0^\tau \int_0^t \int_Q |\nabla \mathbf{w}_\delta(t', x)|^2 dx dt' dt + \int_0^\tau D_\delta(t)\, dt = \\ \displaystyle \frac{1}{2}\tau \int_Q (|\overline{\mathbf{u}_0}(x)|^2 + \delta^2 |\nabla \overline{\mathbf{u}_0}(x)|^2)\, dx, \end{array} \right. \tag{3.6}$$

where one has noted $\overline{\mathbf{u}_0} = A_\delta^{-1}\mathbf{u}_0$. One studies first the term $I_\delta = \delta^2 \int_Q |\nabla \overline{\mathbf{u}_0}(x)|^2 dx \ge 0$. Recall that one has

$$-\delta^2 \Delta \overline{\mathbf{u}_0} + \overline{\mathbf{u}_0} = \mathbf{u}_0. \tag{3.7}$$

Thus, taking $\overline{\mathbf{u}_0}$ as test function yields

$$\delta^2 \int_Q |\nabla \overline{\mathbf{u}_0}(x)|^2 dx + \int_Q |\overline{\mathbf{u}_0}(x)|^2 dx = \int_Q \overline{\mathbf{u}_0}(x).\mathbf{u}_0(x)\, dx. \tag{3.8}$$

By putting $\overline{\mathbf{u}_0} = \mathbf{u}_{0,\delta}$, (3.8) makes sure that the sequence $(\mathbf{u}_{0,\delta})_{\delta>0}$ is bounded in L^2_x. Thus its converges weakly (up to a subsequence) to some $\mathbf{g} \in L^2_x$. One takes a smooth test vector field $\mathbf{v}$ in (3.7) and one computes doing two part integrations on a cell. One has

$$-\delta^2 \int_Q \mathbf{u}_{0,\delta}.\Delta \mathbf{v} + \int_Q \mathbf{u}_{0,\delta}.\mathbf{v} = \int_Q \mathbf{u}_0.\mathbf{v}.$$

When δ goes to zero, one obtains $\displaystyle\int_Q \mathbf{u}_0.\mathbf{v} = \int_Q \mathbf{g}.\mathbf{v}$. Therefore, $\mathbf{g} = \mathbf{u}_0$. The limit being unique, all the sequence converges. One let δ go to zero in (3.8) and therefore

$$\liminf I_\delta + \liminf \int_Q |\mathbf{u}_{0,\delta}|^2 = \int_Q |\mathbf{u}_0|^2 = \limsup I_\delta + \limsup \int_Q |\mathbf{u}_{0,\delta}|^2.$$

On one hand, $\liminf I_\delta \geq 0$. On the other hand, $\displaystyle\liminf \int_Q |\mathbf{u}_{0,\delta}|^2 \geq \int_Q |\mathbf{u}_0|^2$. Consequently, each term being non negative,

$$\liminf I_\delta = \limsup I_\delta = 0 = \lim_{\delta\to 0} \delta^2 \int_Q |\nabla \overline{\mathbf{u}_0}(x)|^2 dx \tag{3.9}$$

Notice that in addition, one has proved that the sequence $(\mathbf{u}_{0,\delta})_{\delta>0}$ converges strongly towards $\mathbf{u}_0$ in L^2_x because of the weak convergence, combined to the convergence of the norms, consequence of the previous argument, that means

$$\lim_{\delta\to 0} \int_Q |\mathbf{u}_{0,\delta}(x)|^2 dx = \int_Q |\mathbf{u}_0(x)|^2 dx. \tag{3.10}$$

Let now consider the term $\displaystyle\int_0^\tau D_\delta(t)\,dt$ in equality (3.6). Notice first that $D_\delta(t) \geq 0$. Each term in the r.h.s of (3.6) being non negative, one can conclude that the sequence $(D_\delta(t))_{\delta>0}$ is bounded in $L^1([0,T])$, T being any non negative real number that one fixes until the end of the proof. Thus, up to a subsequence, it converges weakly in the sense of measures to a non negative Radon measure $\mu_1(t)$ and one has

$$\lim_{\delta\to 0} \int_0^\tau D_\delta(t)\,dt = \int_0^\tau d\mu_1(t). \tag{3.11}$$

Moreover, equality (2.6) combined with (3.9) and (3.10) makes sure that the sequence $(|\nabla \mathbf{w}_\delta|^2)_{\delta>0}$ is bounded in $L^1_{t,x}$. The weak convergence of $(\mathbf{w}_\delta)_{\delta>0}$ towards $\mathbf{u}$ in $L^2_t(H^1_x)$ guaranties the existence of a non negative Radon space periodic defect measure $\mu_2(t,x)$ such that

$$\left\{ \begin{array}{l} \displaystyle\lim_{\delta\to 0} \int_0^\tau \int_0^t \int_Q |\nabla \mathbf{w}_\delta(t',x)|^2 dx dt' dt = \\ \displaystyle\int_0^\tau \int_0^t \int_Q |\nabla \mathbf{u}(t',x)|^2 dx dt' dt + \int_0^\tau \int_0^t \int_Q d\mu_2(t',x). \end{array} \right. \tag{3.12}$$

Finally, by the strong convergence of $(\mathbf{w}_\delta)_{\delta>0}$ towards $\mathbf{u}$ in $L^2_{t,x}$, one has

$$\lim_{\delta\to 0} \int_0^\tau \int_Q |\mathbf{w}_\delta(t,x)|^2 dx dt = \int_0^\tau \int_Q |\mathbf{u}(t,x)|^2 dx dt. \tag{3.13}$$

By putting together (3.9), (3.10), (3.11), (3.12), (3.13), one obtains (for $\tau \in [0,T]$),

$$\int_0^\tau E(t,\mathbf{u})\,dt + \int_0^\tau \int_0^t \int_Q d\mu_2(t',x) + \int_0^\tau d\mu_1(t) = \tau E_0, \tag{3.14}$$

yielding

$$E(t,\mathbf{u}) + \int_0^t \int_Q d\mu_2(t',x) + \mu_1(t) = E_0. \tag{3.15}$$

One deduces (3.3), $E(t,\mathbf{u}) \leq E_0$, thanks to the positiveness of μ_1 and μ_2. Notice that (3.15) does not depend on T and is true for each $t \in I\!R$. Inequalities (3.4) and (3.5) are direct consequences of (3.3). Classical results make sure that $\mathbf{u} \in C^0_t(L^2_{x,w})$. This tells that $\mathbf{u}$ is continuous with respect to the time into L^2_x equipped with its weak topology.

It remains to prove that $t \to E(t, \mathbf{u})$ is a non increasing function of t. Let $t_0 \in I\!R$. As $\mathbf{u} \in C^0_t(L^2_{x,w})$, one have $\mathbf{u}(t_0, x) \in L^2_x$, and $\mathbf{u}(t_0, x) \in V$. Then one can solve (1.1) by replacing $\mathbf{u}_0$ by $\mathbf{u}(t_0, x)$ in (2.4) for $t \geq t_0$. Clearly, when substituting 0 by t_0 one has for $t \geq t_0$, $\mathbf{u} \in \mathcal{V}(\mathbf{u}(t_0, x))$. The previous reasoning applies and then

$$\forall\, t \geq t_0, \quad \frac{1}{2}\int_Q |\mathbf{u}(t,x)|^2 dx + \nu \int_{t_0}^t \int_Q |\nabla \mathbf{u}(t',x)|^2 dx dt' \leq \int_Q |\mathbf{u}(t_0,x)|^2 dx. \tag{3.16}$$

Adding $\nu \int_0^{t_0} \int_Q |\nabla \mathbf{u}(t',x)|^2 dx dt'$ in both side of (3.16) yields $E(t, \mathbf{u}) \leq E(t_0, \mathbf{u})$ for each $t \geq t_0$, which means that $E(t, \mathbf{u})$ is a non increasing function of t, finishing the proof of Theorem 3.1.

Remark 3.1 *Beside the scope of the previous proof is a property of singular perturbation equations due to the operator A_δ for small values of δ. The reader may attempt that boundary layers can appear. But we are here in a periodic case and looking for L^p properties. This class of problems has been studied by J. -L. Lions [17] in the Hilbert case. In our case, one can prove a more general result.*

Lemma 3.1 *Let $\varphi \in L^p_x$, $1 \leq p < \infty$. Then one have*

$$||A_\delta^{-1}\varphi||_{L^p_x} \leq ||\varphi||_{L^p_x}. \tag{3.17}$$

Moreover, when $p > 1$, $(A_\delta^{-1}\varphi)_{\delta>0}$ converges towards φ strongly in L^p_x.

Proof. Put $\overline{\varphi} = A_\delta^{-1}\varphi$. Recall that

$$-\delta^2 \Delta\overline{\varphi} + \overline{\varphi} = \varphi. \tag{3.18}$$

Take $\psi(\overline{\varphi}) = \overline{\varphi}|\overline{\varphi}|^{p-2}$ as test function in (3.18) when $p > 1$, $\psi(\overline{\varphi}) = \mathrm{sgn}(\overline{\varphi})$ when $p = 1$ and integrate by part (eventually use truncations and pass to the limit; we skip here this kind of details, developed in a similar context in part 5). This yields to

$$\delta^2 \int_Q \psi'(\overline{\varphi})|\nabla\overline{\varphi}|^2 + \int_Q |\overline{\varphi}|^p = \int_Q \varphi\psi(\overline{\varphi}). \tag{3.19}$$

Because ψ is a non decreasing function, we can deduce from (3.19) that

$$\int_Q |\overline{\varphi}|^p \leq \int_Q \varphi\psi(\overline{\varphi}). \tag{3.20}$$

Inequality (3.17) is directly deduced from (3.20) when $p = 1$. Assume now that $p > 1$. Then (3.20) yields

$$\int_Q |\overline{\varphi}|^p \leq \int_Q |\varphi|\, |\overline{\varphi}|^{p-1}. \tag{3.21}$$

We use Hölder inequality in the r.h.s of (3.21). Then (3.21) becomes

$$||\overline{\varphi}||^p_{L^p_x} \leq ||\varphi||_{L^p_x} ||\overline{\varphi}||^{p-1}_{L^p_x}, \tag{3.22}$$

yielding (3.17). When $p > 1$, (3.17) tells that from the sequence $(A_\delta^{-1}\varphi)_{\delta>0}$, we can extract a subsequence (still denote by the same) which converges weakly in L^p_x towards some $g \in L^p_x$. In (3.18), take v a smooth test function and integrate by part:

$$-\delta^2 \int_Q \overline{\varphi}\Delta v + \int_Q v\overline{\varphi} = \int_Q v\varphi. \tag{3.23}$$

In (3.23), the term $\int \overline{\varphi}\Delta v$ converges towards $\int g\Delta v$ when δ goes to zero. Thus, when δ goes to zero, one have

$$\int_Q vg = \int_Q v\varphi. \tag{3.24}$$

Then, $g = \varphi$ *a.e.* The space L^p_x being uniformely convex for $p > 1$, we deduce from (3.18) the strong convergence in L^p_x. Finally, by uniqueness of the limit, all the sequence converges and the proof is complete.

3.2 Interpolation

Notice first that by Sobolev imbedding theorem, (3.5) yields

$$||\mathbf{u}||_{L^2_t(L^6_x)} \leq C_{s6}\sqrt{\frac{E_0}{2\nu}}, \tag{3.25}$$

where C_{s6} is the Sobolev constant.

For the sake of the simplicity, one notes for $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$,

$$n_{p,q} = ||\mathbf{u}||_{L^p_t(L^q_x)}.$$

Lemma 3.2 *One has*

$$\forall\, r \in [2,6], \quad n_{\frac{4r}{3(r-2)},r} \leq n_{\infty,2}^{\frac{6-r}{2r}}\, n_{2,6}^{\frac{3(r-2)}{2r}}. \tag{3.26}$$

Corollary 3.2 *The following holds*

$$\forall\, r \in [2,6], \quad n_{\frac{4r}{3(r-2)},r} \leq \frac{C_{s6}^{\frac{3(r-2)}{2r}}}{(2\nu)^{\frac{3(r-2)}{4r}}}\sqrt{E_0}. \tag{3.27}$$

Corollary 3.2 and in particular (3.27) follows from (3.26) combined with (3.25), (3.4) and (3.5)

Proof of lemma 3.2. Let $r \in [2,6]$ and write $r = 2\theta + 6(1-\theta)$. By Hölder inequality one has

$$\int_Q |\mathbf{u}|^r \leq \int_Q \left(|\mathbf{u}|^2\right)^\theta \left(\int_Q |\mathbf{u}|^6\right)^{1-\theta} \leq n_{\infty,2}^{2\theta}||\mathbf{u}||_{L^6_x}^{6(1-\theta)}. \tag{3.28}$$

Writing $\theta = \frac{6-r}{4}$ yields

$$\left(\int_Q |\mathbf{u}|^r\right)^{\frac{4}{3(r-2)}} \leq n_{\infty,2}^{\frac{2(6-r)}{3(r-2)}}||\mathbf{u}||_{L^6_x}^2, \tag{3.29}$$

that is

$$||\mathbf{u}||_{L^r_x}^{\frac{4r}{3(r-2)}} \leq n_{\infty,2}^{\frac{2(6-r)}{3(r-2)}}||\mathbf{u}||_{L^6_x}^2. \tag{3.30}$$

Inequality (3.26) is deduced from (3.30) after integrating with respect to the time and an easy algebraic calculation. In what follows, one puts

$$t(r) = \frac{4r}{3(r-2)}, \quad r \in [2,6]. \tag{3.31}$$

3.3 Regularity of the convective term

Lemma 3.3 *The convective term satisfies*

$$\forall\, r \in [2,6], \quad ||(\mathbf{u}\nabla)\mathbf{u}||_{L_t^{\frac{4r}{5r-6}}(L_x^{\frac{2r}{r+2}})} \le C_{s6}^{\frac{3(r-2)}{2r}} (2\nu)^{\frac{6-5r}{4r}} E_0. \tag{3.32}$$

Proof of corollary 3.3. An easy computation combined with Hölder inequality yields

$$||(\mathbf{u}\nabla)\mathbf{u}||_{L_t^{\frac{4r}{5r-6}}(L_x^{\frac{2r}{r+2}})} \le ||\mathbf{u}||_{L_t^{t(r)}(L_x^r)} ||\nabla\mathbf{u}||_{L_t^2(L_x^2)}.$$

Then (3.32) is a consequence of (3.5) combined to (3.27). Notice that in particular

$$(\mathbf{u}\nabla)\mathbf{u} \in L_t^1(L_x^{3/2}) \cap L_t^2(L_x^1).$$

4 On the equation for the pressure

4.1 Orientations

Let $\mathbf{u}_0 \in V$, $(\mathbf{u}, p) \in \mathcal{V}(\mathbf{u}_0) \times \mathcal{P}(\mathbf{u}_0)$. We start by classical estimates on p.

Next, because $(\mathbf{u}, p)$ is a LES solution, it is limit of a sequence $(\mathbf{w}_n, q_n)_{n\in I\!N}$ such that $(\mathbf{w}_n, q_n)$ is the unique solution to

$$\begin{cases} \partial_t \mathbf{w}_n + \nabla \cdot (A_{\delta_n}^{-1}(\mathbf{w}_n\mathbf{w}_n)) - \nu\Delta\mathbf{w}_n + \nabla q_n = 0, \\ \nabla \cdot \mathbf{w}_n = 0, \\ (\mathbf{w}_n)_{t=0} = A_{\delta_n}^{-1}\mathbf{u}_0, \end{cases} \tag{4.1}$$

where $(\delta_n)_{n\in I\!N}$ is a sequence of non negative numbers which converges to 0.

We seek in this section for more estimates for q_n when n is fixed. Next, we derive entropy estimates satisfied by the sequence $(q_n)_{n\in I\!N}$.

4.2 Direct estimates

We start with direct estimates on the pressure. Taking the divergence of the motion's equation in (1.1) yields the following equation for the pressure

$$-\Delta p = -\nabla \cdot ((\mathbf{u}\nabla)\mathbf{u}). \tag{4.2}$$

Lemma 4.1 *One has*

$$\forall\, r \in [2,6], \quad ||p||_{L_t^{\frac{4r}{5r-6}}(W_x^{1,\frac{2r}{r+2}})} \le C_{s6}^{\frac{3(r-2)}{2r}} (2\nu)^{\frac{6-5r}{4r}} E_0. \tag{4.3}$$

Proof This is a direct consequence of the classical elliptic theory combined with (3.32) and the fact that we are working with periodic conditions. Notice that one has in particular

$$p \in L_t^1(W_x^{1,3/2}). \tag{4.4}$$

Remark 4.1 *Such kind of estimates for the pressure can be founded in a book of P. -L. Lions [20]. We mention that improved estimates using Hardy spaces can be found in [6].*

Remark 4.2 *Beside de scope of estimate* (4.3) *we have used the fact that* $-\Delta$ *is an isomorphism between* $W_x^{1,p}$ *and* $W_x^{-1,p}$*, a fact that we shall use again in the remainder. This is mainly due to the famous well known regularity results of S. Agmon, A. Douglis and L. Nirenberg [1]. We mention also a proof of a similar result due to J. -L. Lions and E. Magenes [19].*

4.3 Improved estimates for the approximations

One give here an improvement to the results of [12] needed for future applications in the paper. Fix n and denote by q_n the corresponding pressure in (2.4).
One first remark that for any non compressible field $\mathbf{v}$ (with $\nabla \cdot \mathbf{v} = 0$), one has

$$\nabla \cdot ((\mathbf{v}\nabla)\mathbf{v}) = \nabla \mathbf{v} : \nabla \mathbf{v}^t. \tag{4.5}$$

Therefore, equation (4.2) can be rewrited as

$$-\Delta p = -\nabla \mathbf{u} : \nabla \mathbf{u}^t \in L^1_{t,x}. \tag{4.6}$$

Lemma 4.2 *For a fixed n one have*

$$q_n \in L^\infty_t(W^{4,3}_x). \tag{4.7}$$

Proof. Thanks to (4.5) and (2.3), the equation for q_n can be written under the form

$$-\Delta q_n = A^{-1}_{\delta_n}(\nabla \mathbf{w}_n : \nabla \mathbf{w}^t_n). \tag{4.8}$$

Due to the regularity result (2.7), $\nabla \mathbf{w}_n \in L^\infty_t(H^1_x)$ and thus $\nabla \mathbf{w}_n : \nabla \mathbf{w}^t_n \in L^\infty_t(L^3_x)$. Consequently, $A^{-1}_{\delta_n}(\nabla \mathbf{w}_n : \nabla \mathbf{w}^t_n) \in L^\infty_t(W^{2,3}_x)$ and (4.7) is obvious.

When one combines (2.7) with (4.7), one obtains for a fixed n,

$$\partial_t \mathbf{w}_n \in L^2_t(H^1_x) \cap L^\infty_t(L^2_x). \tag{4.9}$$

4.4 Entropy estimates

Let g be a bounded Lipchitz function with a derivative having a finite number of discontinuities in view of using the results of Stampacchia [24]. We assume that

$$(g')^2 = g', \quad g'' = 0. \tag{4.10}$$

Lemma 4.3 *One has*

$$||g(q_n)||_{L^2_t(H^1_x)} \leq C||g||_\infty \frac{E_0}{2\nu}, \tag{4.11}$$

where C is a generic constant.

Proof. Thanks to the regularity property (4.7), all the manipulations below are justified, thanks to [24]. Take $A_{\delta_n} q_n = -\delta_n^2 \Delta g(q_n) + g(q_n)$ as test function in (4.8). Notice that $\Delta g(q_n) = g'(q_n)\Delta q_n$ because of (4.10). Using the fact that A_δ is self-adjoint, this leads to

$$\int_Q g'(q_n)|\nabla q_n|^2 + \delta_n^2 \int_Q g'(q_n)|\Delta q_n|^2 = \int_Q (\nabla \mathbf{w}_n : \nabla \mathbf{w}^t_n)\, g(q_n). \tag{4.12}$$

Because of (4.10),

$$\int_Q g'(q_n)|\Delta q_n|^2 = \int_Q |g'(q_n)\Delta q_n|^2 = \int_Q |\Delta g(q_n)|^2 \geq 0.$$

in the same way

$$\int_Q g'(q_n)|\nabla q_n|^2 = \int_Q |g'(q_n)\nabla q_n|^2 = \int_Q |\nabla g(q_n)|^2.$$

Finally,

$$\left|\int_Q (\nabla \mathbf{w}_n : \nabla \mathbf{w}_n^t)\, g(q_n)\right| \leq ||g||_\infty ||\nabla \mathbf{w}_n||^2_{L^2_x}.$$

Therefore, (4.12) yields

$$\int_Q |\nabla g(q_n)|^2 \leq ||g||_\infty ||\nabla \mathbf{w}_n||^2_{L^2_x}. \tag{4.13}$$

Estimate (4.11) is obtained when one integrates (4.13) with respect to the time and one uses (2.6), (3.9) and (3.10).

Remark 4.3 *When one takes $g = T_k$ (which satisfies* (4.10)*), one sees that the sequence $(T_k q_n)_{n\in I\!N}$ is bounded in $L^2_t(H^1_x)$. Therefore, one can extract from this sequence a subsequence which converges weakly in $L^2_t(H^1_x)$ to a function $\tilde{p}_k$. Arguing as in [22], we can prove that there is a measurable function $\tilde{p}$ such that $\tilde{p}_k = T_k(\tilde{p})$. Despite the fact that $(q_n)_{n\in I\!N}$ converges weakly to p in $L^{4/3}_{t,x}$, we have no reason to claim that $p = \tilde{p}$. Indeed, if the sequence $(q_n)_{n\in I\!N}$ devellops high oscillations at infinity (what it probably does), then one can have $p \neq \tilde{p}$.*

5 The vorticity equation

5.1 Statement of the main result

Let $\mathbf{u}_0 \in V$, $\mathbf{u} \in \mathcal{V}(\mathbf{u}_0)$. One defines the vorticity by

$$\omega = \nabla \times \mathbf{u}. \tag{5.1}$$

Notice that one deduces from estimate (3.5) the natural estimate for the vorticity

$$||\omega||_{L^2_t(L^2_x)} \leq C\sqrt{\frac{E_0}{2\nu}}, \tag{5.2}$$

C being a generic constant.

Recall the classical formula for incompressible vector fields $\mathbf{v}$,

$$\nabla \times \nabla \cdot (\mathbf{v}\mathbf{v}) = (\mathbf{v}\nabla)(\nabla \times \mathbf{v}) - ((\nabla \times \mathbf{v})\nabla)\mathbf{v}. \tag{5.3}$$

Taking formally the *curl* of the motion equation in (1.1) yields the equation for ω (see for instance in [3]):

$$\begin{cases} \partial_t \omega + (\mathbf{u}\nabla)\omega - \nu\Delta\omega = (\omega\nabla)\mathbf{u}, \\ \omega_{t=0} = \omega_0 = \nabla \times \mathbf{u}_0. \end{cases} \tag{5.4}$$

We shall focus our attention on the following equation:

$$\begin{cases} \partial_t \omega + (\mathbf{u}\nabla)\omega - \nu\Delta\omega = ((\nabla \times \mathbf{u})\nabla)\mathbf{u}, \\ \omega_{t=0} = \omega_0 = \nabla \times \mathbf{u}_0, \end{cases} \tag{5.5}$$

where ω is space periodic and have a zero mean value. We do not know if for any $\mathbf{u}_0 \in V$ and $\mathbf{u} \in \mathcal{V}(\mathbf{u}_0)$ and a suitable assumption on ω_0, equation (5.5) admits a unique distributional solution ω which satisfies estimates (5.2) and is also solution to equation (5.4).

We assume throughout this section

$$\omega_0 = \nabla \times \mathbf{u}_0 \in L^1_x. \tag{5.6}$$

We first remark that (3.5) makes sure that

$$((\nabla \times \mathbf{u})\nabla)\mathbf{u} \in L^1_{t,x} \tag{5.7}$$

Then, (5.5) is an equation with "*a second hand side in* L^1".

Until the end of this part, one shall have proved the following.

Theorem 5.1 *Let* $\mathbf{u}_0 \in V$ *satisfying* (5.6)*, that is* $\nabla \times \mathbf{u}_0 \in L^1_x$*, and* $\mathbf{u} \in \mathcal{V}(\mathbf{u}_0)$*. Then* $\omega = \nabla \times \mathbf{u}$ *is a distributional solution to equation* (5.5)*, hence to equation* (5.4)*, which satisfies*

$$\omega \in L^\infty_t(L^1_x), \tag{5.8}$$

$$\forall\, k \in I\!R^+, \quad T_k(\omega) \in L^2_t(H^1_x), \tag{5.9}$$

$$\omega \in \bigcap_{p<5/4} L^p_t(W^{1,p}). \tag{5.10}$$

Moreover, there exists a constant C *such that*

$$||\omega||_{L^\infty_t(L^1_x)} \leq C\left(\frac{E_0}{2\nu} + ||\omega_0||_{L^1_x}\right). \tag{5.11}$$

In the previous statement, T_k denotes the truncation function at height k, that is

$$T_k(x) = x \;\text{ if }\; |x| \leq k, \quad T_k(x) = k\frac{x}{|x|} \;\text{ if }\; |x| \geq k. \tag{5.12}$$

If $\omega = (\omega^1, \omega^2, \omega^3)$, $T_k(\omega)$ is the vector field $(T_k(\omega^1), T_k(\omega^2), T_k(\omega^3))$.

Remark 5.1 *The regularity result* (5.10) *is formally a consequence of the Boccardo-Gallouët inequalities [5] that we have carfully to establish below.*

Remark 5.2 *As already mentionned, similar estimate as* (5.11) *can be found in [7] and in [20]. It is also mentionned in the review of C. Bardos and B. Nicolaenko [2].*

In what follows, we shall note H the space defined by

$$\mathrm{H} = \{\mathbf{v} \in V;\, \nabla \times \mathbf{v} \in L^1_x\}. \tag{5.13}$$

5.2 Approximations for the vorticity's equation

Fix $\mathbf{u}_0 \in \mathrm{H}$, where H is defined by (5.13). Consider $(\mathbf{u}, p) \in \mathcal{V}(\mathbf{u}_0) \times \mathcal{P}(\mathbf{u}_0)$. Thus it is a limit of a sequence $(\mathbf{w}_n, q_n)_{n\in I\!N}$ solution of

$$\begin{cases} \partial_t \mathbf{w}_n + \nabla \cdot (A^{-1}_{\delta_n}(\mathbf{w}_n\mathbf{w}_n)) - \nu\Delta\mathbf{w}_n + \nabla q_n = 0, \\ \nabla \cdot \mathbf{w}_n = 0, \\ (\mathbf{w}_n)_{t=0} = A^{-1}_{\delta_n}\mathbf{u}_0, \end{cases} \tag{5.14}$$

where δ_n goes to zero when n goes to infinity. Let

$$\omega_n = \nabla \times \mathbf{w}_n. \tag{5.15}$$

Using (2.3) and (5.3), we know that ω_n satisfies the equation

$$\left\{ \begin{array}{l} \partial_t \omega_n + A_{\delta_n}^{-1}((\mathbf{w}_n \nabla)\omega_n) - \nu\Delta\omega_n = A_{\delta_n}^{-1}(((\nabla \times \mathbf{w}_n)\nabla)\mathbf{w}_n), \\ (\omega_n)_{t=0} = A_{\delta_n}^{-1}(\omega_0) = \nabla \times (A_{\delta_n}^{-1}\mathbf{u}_0). \end{array} \right. \tag{5.16}$$

Notice that by (2.7) and (4.9), we know that for fixed n, $\mathbf{w}_n \in L_t^2(H_x^3) \cap L_t^\infty(H_x^2)$, $\partial_t \mathbf{w} \in L_t^2(H_x^1)$. Thus one has for a fixed n,

$$\omega_n \in L_t^2(H_x^2) \cap L_t^\infty(H_x^1), \tag{5.17}$$

$$\partial_t \omega_n \in L_t^2(L_x^2). \tag{5.18}$$

Lemma 5.1 *The sequence $(\omega_n)_{n\in I\!N}$ is bounded in $L_t^\infty(L_x^1)$ and there exists a constant C such that*

$$||\omega_n||_{L_t^\infty(L_x^1)} \leq C\zeta_n, \tag{5.19}$$

where $\displaystyle\lim_{n\to\infty} \zeta_n = \frac{E_0}{2\nu} + ||\omega_0||_{L_x^1}$.

Proof. Put $\omega_n = (\omega_n^1, \omega_n^2, \omega_n^3)$. Writing (5.16) component by component yields

$$\partial_t \omega_n^j + A_{\delta_n}^{-1}((\mathbf{w}_n \nabla)\omega_n^j) - \nu\Delta\omega_n^j = A_{\delta_n}^{-1}((\nabla \times \mathbf{w}_n)^i \partial_i \mathbf{w}_n^j). \tag{5.20}$$

Let $\varepsilon > 0$ and φ_ε the function defined on $I\!R$ by

$$\forall\, 0 \leq x \leq \varepsilon,\ \varphi_\varepsilon(x) = x, \quad \forall\, x \geq \varepsilon,\ \varphi_\varepsilon(x) = 1, \quad \varphi_\varepsilon(-x) = -\varphi_\varepsilon(x).$$

Let

$$\psi_\varepsilon(x) = \int_0^x \varphi_\varepsilon(x')\, dx'.$$

Take $A_{\delta_n}\varphi_\varepsilon(\omega_n^j) = -\delta^2\Delta\varphi_\varepsilon(\omega_n^j) + \varphi_\varepsilon(\omega_n^j)$ as test function in (5.16) and integrate by part. This operation makes sense thanks to the regularity properties (5.17) and (5.18). One obtains:

$$\left\{ \begin{array}{l} \displaystyle\frac{d}{dt}\int_Q \psi_\varepsilon(\omega_n^j) - \delta_n^2 \int_Q \varphi_\varepsilon'(\omega_n^j)\partial_t\omega_n^j \Delta\omega_n^j + \int_Q A_{\delta_n}^{-1}((\mathbf{w}_n\nabla)\omega_n^j)A_{\delta_n}\varphi_\varepsilon(\omega_n^j) + \\ \displaystyle\nu\int_Q \varphi_\varepsilon'(\omega_n^j)|\nabla\omega_n^j|^2 + \nu\delta_n^2\int_Q \varphi_\varepsilon'(\omega_n^j)|\Delta\omega_n^j|^2 = \\ \displaystyle\int_Q A_{\delta_n}^{-1}((\nabla\times\mathbf{w}_n)^i\partial_i\mathbf{w}_n^j)A_{\delta_n}\varphi_\varepsilon(\omega_n^j). \end{array} \right. \tag{5.21}$$

We have used the fact that $\Delta\varphi_\varepsilon(\omega_n^j) = \varphi_\varepsilon'(\omega_n^j)\omega_n^j + \varphi_\varepsilon''(\omega_n^j)|\nabla\omega_n^j|^2$ and $\varphi_\varepsilon'' = 0$. These computations are justified by the results of G. Stampacchia [24] combined with (5.17) and (5.18).

Because A_{δ_n} is self-adjoint and $\mathbf{w}_n$ has a zero divergence,

$$\int_Q A_{\delta_n}^{-1}((\mathbf{w}_n\nabla)\omega_n^j)A_{\delta_n}\varphi_\varepsilon(\omega_n^j) = \int_Q ((\mathbf{w}_n\nabla)\omega_n^j)\varphi_\varepsilon(\omega_n^j) = 0. \tag{5.22}$$

Moreover, because $|\varphi_\varepsilon| \leq 1$, one has

$$\left\{ \begin{array}{l} \displaystyle\left|\int_Q A_{\delta_n}^{-1}(((\nabla\times\mathbf{w}_n)^i\partial_i\mathbf{w}_n^j)A_{\delta_n}\varphi_\varepsilon(\omega_n^j)\right| = \\ \displaystyle\left|\int_Q ((\nabla\times\mathbf{w}_n)^i\partial_i\mathbf{w}_n^j)\varphi_\varepsilon(\omega_n^j)\right| \leq C||\nabla\mathbf{w}_n||_{L_x^2}^2, \end{array} \right. \tag{5.23}$$

where C denotes a generic constant. Finally, one has

$$\int_Q \varphi'_\varepsilon(\omega_n^j)|\nabla\omega_n^j|^2 \geq 0, \quad \int_0^t \int_Q \varphi'_\varepsilon(\omega_n^j)|\Delta\omega_n^j|^2 \geq 0. \tag{5.24}$$

When one combines (5.21), (5.22), (5.23) and (5.24), one obtains

$$\frac{d}{dt}\int_Q \psi_\varepsilon(\omega_n^j) - \delta_n^2 \int_Q \varphi'_\varepsilon(\omega_n^j)\partial_t\omega_n^j\Delta\omega_n^j \leq C||\nabla\mathbf{w}_n||^2_{L^2_x}. \tag{5.25}$$

One integrates (5.25) with respect to the time on the interval $[0,t]$:

$$\int_Q \psi_\varepsilon(\omega_n^j(t,x))\,dx - \delta_n^2\int_0^t\int_Q \varphi'_\varepsilon(\omega_n^j)\partial_t\omega_n^j\Delta\omega_n^j \leq C||\nabla\mathbf{w}_n||^2_{L^2_x} + \int_Q \psi_\varepsilon((\omega_n^j)_0). \tag{5.26}$$

Note that it is easy seen that $\omega_n^j \in C^0_t(L^2_x)$ thanks to (5.18). Then because

$$\lim_{\varepsilon\to 0}\psi_\varepsilon(x) = |x|,$$

it is obvious that

$$\lim_{\varepsilon\to 0}\int_Q \psi_\varepsilon(\omega_n^j(t,x))\,dx = \int_Q |\omega_n^j(t,x)|\,dx, \quad \lim_{\varepsilon\to 0}\int_Q \psi_\varepsilon((\omega_n^j)_0)\,dx = ||(\omega_n^j)_0||_{L^1_x}. \tag{5.27}$$

Consider

$$E_n(t) = \left\{(t',x)\in[0,t]\times Q;\ \omega_n^j(t',x) = 0\right\}.$$

We note that

$$\lim_{\varepsilon\to 0}\varphi'_\varepsilon(\omega_n^j) = 1\!\!|_{E_n(t)} \quad \text{a.e. in } \ [0,t]\times Q,$$

where $1\!\!|_{E_n(t)}(x) = 0$ if $x \notin E_n(t)$, $1\!\!|_{E_n(t)}(x) = 1$ if $x \in E_n(t)$. Because $|\varphi'_\varepsilon| \leq 1$ and thanks to (5.17) and (5.18), Lebesgue's Theorem yields

$$\lim_{\varepsilon\to 0}\int_0^t\int_Q \varphi'_\varepsilon(\omega_n^j)\partial_t\omega_n^j\Delta\omega_n^j = \int\int_{E_n(t)} \partial_t\omega_n^j\Delta\omega_n^j. \tag{5.28}$$

Using again the results of G. Stampacchia [24] combined to (5.17) and (5.18), one has

$$\int\int_{E_n(t)} \partial_t\omega_n^j\Delta\omega_n^j = 0. \tag{5.29}$$

When one combines (5.26), (5.27), (5.28) and (5.29), one obtains

$$\int_Q |\omega_n^j(t,x)|\,dx \leq C||\nabla\mathbf{w}_n||^2_{L^2_t(L^2_x)} + ||(\omega_n^j)_0||_{L^1_x}. \tag{5.30}$$

Then (5.19) is a consequence of (5.30) combined to (2.6), (3.9) and (3.10). The proof of Lemma 5.1 is complete.

5.3 Entropy inequalities

The goal of this part is the proof of general entropy inequalities in order to prove (5.9) and (5.10) for the sequence $(\omega_n)_{n\in I\!N}$.

Let g be a Lipschitz bounded function such that its derivative has a finite number of discontinuities. Let

$$G(x) = \int_0^x g(x')\,dx'.$$

Assume that

(5.31) $$(g')^2 = g', \quad g'' = 0.$$

We shall note in the following

(5.32) $$B^k_{n,j} = \Big\{(t,x) \in I\!R \times Q;\; k \le |\omega^j_n(t,x)| \le k+1\Big\}.$$

Lemma 5.2 *The sequence $(\omega^j_n)_{n \in I\!N}$ satisfies for all t*

(5.33) $$\begin{cases} \displaystyle\int_Q G(\omega^j_n(t,x))\,dx \;+\delta_n^2 \int_Q |\nabla g(\omega^j_n(t,x))|^2 dx + \\ \displaystyle\nu \int_0^t \int_Q g'(\omega^j_n(t',x))|\nabla \omega^j_n(t',x)|^2 dx dt' + \nu\delta_n^2 \int_0^t \int_Q |\Delta g(\omega^j_n(t,x))|^2 dx dt' \le \\ \displaystyle C||g||_\infty E_0 + \int_Q G(\overline{\omega_0}^j(x))\,dx + \delta_n^2 \int_Q |\nabla g(\overline{\omega_0}^j(x)|^2 dx, \end{cases}$$

where $\overline{\omega_0}^j = {A_{\delta_n}}^{-1} {\omega_0}^j$.

Lemma 5.3 *One has*

(5.34) $$\delta_n^2 \int_Q |\nabla g(\overline{\omega_0}^j(x)|^2 dx \le 2||g||_\infty ||\omega^j_0||_{L^1_x}.$$

Corollary 5.1 *The following inequalities hold*

(5.35) $$\forall\, k > 0, \quad ||T_k(\omega^j_n)||^2_{L^2_x(H^1_x)} \le C k E_0 + k^2 mes(Q) + (2k+1)||\omega^j_0||_{L^1_x},$$

(5.36) $$\forall\, k > 0, \quad \int\int_{B^k_{n,j}} |\nabla \omega^j_n|^2 \le C E_0 + 2||\omega^j_0||_{L^1_x} + mes(Q),$$

(5.37) $$\forall\, p < 5/4, \quad ||\omega^j_n||_{L^p_t(W^{1,p}_x)} \le C_p(E_0, ||\omega^j_0||_{L^1_x}),$$

where $C_p(E_0, ||\omega^j_0||_{L^1_x})$ goes to infinity when p goes to $5/4$.

Proof of Lemma 5.2. Due to the regularity results (5.17) and (5.18) for a fixed n, the fact that g is Lipchitz with a derivative with a finite number of discontinuity, the results of G. Stampacchia [24] apply and validate all the following manipulations below.

In equation (5.20) take $A_{\delta_n} g(\omega^j_n) = -\delta_n^2 \Delta g(\omega^j_n) + g(\omega^j_n)$ as test function. Notice that by assumption (5.31), $\Delta g(\omega^j_n) = g'(\omega^j_n)\Delta\omega^j_n$. Moreover, as we have several used before by the fact that A_{δ_n} is self-adjoint,

(5.38) $$\int_Q A^{-1}_{\delta_n}((\nabla \times \mathbf{w}_n)^i \partial_i \mathbf{w}^j_n) A_{\delta_n} g(\omega^j_n) = \int_Q (\nabla \times \mathbf{w}_n)^i \partial_i \mathbf{w}^j_n g(\omega^j_n).$$

Finally, because $\nabla \cdot \mathbf{w}_n = 0$,

(5.39) $$\int_Q A^{-1}_{\delta_n}((\mathbf{w}_n \nabla)\omega^j_n) A_{\delta_n} g(\omega^j_n) = \int_Q (\mathbf{w}_n \nabla)\omega^j_n g(\omega^j_n) = 0.$$

Then one obtains

(5.40) $$\begin{cases} \displaystyle\frac{d}{dt}\int_Q G(\omega^j_n) - \delta_n^2 \int_Q g'(\omega^j_n)\partial_t \omega^j_n \Delta\omega^j_n + \nu \int_Q g'(\omega^j_n)|\nabla\omega^j_n|^2 + \\ \displaystyle\delta_n^2 \nu \int_Q g'(\omega^j_n)|\Delta(\omega^j_n)|^2 \le ||g||_\infty ||\nabla \mathbf{w}_n||^2_{L^2_x}. \end{cases}$$

By using assumption (5.31), $g' = (g')^2$, one has

$$
(5.41) \qquad \begin{cases} \displaystyle\int_Q g'(\omega_n^j)\,\partial_t\omega_n^j\,\Delta\omega_n^j = \int_Q g'(\omega_n^j)\,\partial_t\omega_n^j\, g'(\omega_n^j)\Delta\omega_n^j \;= \\ \displaystyle\int_Q g'(\omega_n^j)\,\partial_t\omega_n^j\,\Delta g(\omega_n^j) = \int_Q \partial_t g(\omega_n^j)\,\Delta g(\omega_n^j) = -\frac{d}{dt}\int_Q |\nabla g(\omega_n^j)|^2. \end{cases}
$$

Still using (5.31),

$$
(5.42) \qquad \int_Q g'(\omega_n^j)|\Delta(\omega_n^j)|^2 = \int_Q g'(\omega_n^j)\Delta(\omega_n^j)g'(\omega_n^j)\Delta(\omega_n^j) = \int_Q |\Delta(g(\omega_n^j))|^2.
$$

When one combines (5.40), (5.41) and (5.42) one obtains

$$
(5.43) \qquad \begin{cases} \displaystyle\frac{d}{dt}\int_Q G(\omega_n^j) + \delta_n^2\frac{d}{dt}\int_Q |\nabla g(\omega_n^j)|^2 + \nu\int_Q g'(\omega_n^j)|\nabla\omega_n^j|^2 + \\ \displaystyle\delta_n^2\nu\int_Q |\Delta(g(\omega_n^j))|^2 \le ||g||_\infty ||\nabla \mathbf{w}_n||^2_{L^2_x}. \end{cases}
$$

Inequality (5.33) follows by integrating (5.37) with respect to the time and by using (2.6), (3.9) and (3.10).

Proof of Lemma 5.3. Recall that

$$
(5.44) \qquad -\delta_n^2\Delta\overline{\omega_0}^j + \overline{\omega_0}^j = \omega_0^j.
$$

Taking $g(\overline{\omega_0}^j)$ as test function in (5.44) yield

$$
(5.45) \qquad \delta_n^2\int_Q g'(\overline{\omega_0}^j)|\nabla\overline{\omega_0}^j|^2 + \int_Q g(\overline{\omega_0}^j)\overline{\omega_0}^j = \int_Q g(\overline{\omega_0}^j)\omega_0^j
$$

By still using (5.31), (5.45) reads

$$
(5.46) \qquad \delta_n^2\int_Q |\nabla g(\overline{\omega_0}^j)|^2 \le ||g||_\infty \left(\int_Q |\overline{\omega_0}^j| + \int_Q |\omega_0^j|\right).
$$

Then (5.34) is a consequence of (5.46) combined to (3.17) in Lemma 3.1.

Remark 5.3 *We do not know if*

$$
\lim_{n\to\infty}\delta_n^2\int_Q |\nabla g(A_{\delta_n}^{-1}{\omega_0}^j)|^2 = 0.
$$

Proof of Corollary 5.1. Inequality (5.35) follows from (5.33) by taking $g(x) = T_k(x)$. Inequality (5.36) is obtained when one takes the odd function $g = g_k$ be such that $g_k(x) = 0$ for $0 \le x \le k$, $g_k(x) = x - k$ for $k \le x \le k+1$ and $g_k(x) = 1$ for $x \ge k+1$. Notice that both T_k and g_k are bounded Lipchitz functions with their derivative having a finite number of discontinuities and satisfy (5.31). The inequalities are consequences of (5.19), (5.34), (3.17) and the reasonning made in [16] page 131 and 132. Finally, (5.37) is a consequence of (5.35) and (5.36) combined with the Boccardo-Gallouët inequality (see in [5]).

5.4 Passing to the limit : proof of theorem 5.1

We now have to prove Theorem 5.1. Therefore, we have to pass to the limit in equation (5.16) to prove that ω is a distributional solution to (5.5) and show that (5.8), (5.9) and (5.10) hold.

Thanks to (5.37) and arguing like in [16], one can extract from the sequence $(\omega_n)_n$ a subsequence (still denoted by the same) such that $(\omega_n)_n$ converges weakly to some $\tilde{\omega}$ in $L^p_t(W^{1,p}_x)$ for all $p < 5/4$. Because $\omega_n = \nabla \times \mathbf{w}_n$, we also know that $(\omega_n)_{n\in I\!N}$ converges weakly to ω in $L^2_{t,x}$. Thus $\tilde{\omega} = \omega$. In particular (5.10) is proved. Moreover, $(-\Delta\omega_n)_{n\in I\!N}$ converges weakly to $-\Delta\omega$ in $L^p(W^{-1,p})$ for each $p < 5/4$.

Let consider now the transport term. For φ a smooth test vector field, one has

$$< A^{-1}_{\delta}((\mathbf{w}_n\nabla)\omega_n), \varphi >= \int_Q \nabla \cdot A^{-1}_{\delta_n}(\mathbf{w}_n\omega_n)\varphi = -\int_Q A^{-1}_{\delta_n}(\mathbf{w}_n\omega_n)\nabla\varphi.$$

Combining (3.17), (3.27) and (5.2), one remarks that the sequence $(A^{-1}_{\delta_n}(\mathbf{w}_n\omega_n))_{n\in I\!N}$ is bounded in $L^{8/7}_t(L^{3/4}_x)$. Thus, $(A^{-1}_{\delta_n}((\mathbf{w}_n\nabla)\omega_n))_{n\in I\!N}$ is bounded in $L^{8/7}_t(W^{-1,3/4}_x)$. Let $\varepsilon > 0$. Because of the strong compactness of $(\mathbf{w}_n)_{n\in I\!N}$ in $L^{8/3-\varepsilon}_t(L^{4-\varepsilon}_x)$ and the weak compactness of $(\omega_n)_{n\in I\!N}$, it is easy seen that $(A^{-1}_{\delta_n}((\mathbf{w}_n\nabla)\omega_n))_{n\in I\!N}$ converges weakly in $L^{8/7-\varepsilon}_t(W^{-1,3/4+\varepsilon}_x)$ to $(\mathbf{u}\nabla)\omega$.

Thanks to the $L^1_{t,x}$ bound of $(A^{-1}_{\delta_n}(((\nabla\times\mathbf{w}_n)\nabla)\mathbf{w}_n))_{n\in I\!N}$, we deduce from the considerations above that $(\partial_t\omega_n)_{n\in I\!N}$ is bounded in $L^1(W^{-3,p})$ for some $p > 1$. Therefore, by using an Aubin-Lions lemma adapted to the L^1 time case (see in [16]), we know that $(\omega_n)_{n\in I\!N}$ is compact in $L^1_{t,x}$. Thererefore, from $(\omega_n)_{n\in I\!N}$ one can extract a subsequence (still denoted by the same) which converges *a.e.* in space-time to ω a and with its modulus dominated by a $L^1_{t,x}$-function.

Now, there exists sets $A_t \subset I\!R$ and $A_x \subset Q$, with $mes(A^c_x) = mes(A^c_t) = 0$ and for all $(t,x) \in A_t \times A_x$, $(\omega_n(t,x))_{n\in I\!N}$ converges to $\omega(t,x)$. Therefore, when one fixes $t \in A_t$, $(\omega_n(t,\cdot))_{n\in I\!N}$ converges a.e. to $\omega(t,\cdot)$. By Fatou's Lemma combined with (5.19) one has for almost every time t,

$$\int_Q |\omega(t,\cdot)| \leq \liminf_{n\in I\!N} \int_Q |\omega_n(t,\cdot)| \leq C\left(\frac{E_0}{2\nu} + ||\omega_0||_{L^1_x}\right).$$

Thereore, $\omega \in L^\infty_t(L^1_x)$ and one has

$$||\omega||_{L^\infty_t(L^1_x)} \leq C\left(\frac{E_0}{2\nu} + ||\omega_0||_{L^1_x}\right). \tag{5.47}$$

Then (5.8) is proved.

Thanks to (5.35), one knows that for a fixed $k > 0$, the sequence $(T_k(\omega_n))_{n\in I\!N}$ is bounded in $L^2_t(H^1_x)$. Thus from this sequence one can extract a subsequence wich weakly converges in $L^2_t(H^1_x)$ to a vector field $\tilde{\omega}_k \in L^2_t(H^1_x)$. But, one also know from the above arguments that $(T_k(\omega_n))_{n\in I\!N}$ converges *a.e.* to $T_k\omega$, and therefore thanks to Lebesgue's Theorem, strongly in $L^2_{x,t}$ ($|T_k(\omega_n)| \leq k \in L^\infty(I\!R\times Q)$). Therefore, $\tilde{\omega}_k = T_k(\omega) \in L^2_t(H^1_x)$ and (5.9) is proved.

It is now easy seen that $(\partial_t\omega_n)_{n\in I\!N}$ converges to $\partial_t\omega$ in the distributional sens. It remains to treat the source term in the equations.

Remark now that for φ a smooth test vector field, one has

$$< A_{\delta_n}^{-1}(((\nabla\times\mathbf{w}_n)\nabla)\mathbf{w}_n),\varphi >= \int_Q \nabla\cdot A_{\delta_n}^{-1}((\nabla\times\mathbf{w}_n)\mathbf{w}_n).\varphi = -\int_Q A_{\delta_n}^{-1}((\nabla\times\mathbf{w}_n)\mathbf{w}_n):\nabla\varphi.$$

Thus, the same argument as above applies, and one concludes that $(A_{\delta_n}^{-1}(((\nabla\times\mathbf{w}_n)\nabla)\mathbf{w}_n))_{n\in I\!N}$ converges to $(\nabla\times\mathbf{u})\nabla)\mathbf{u} = (\omega\nabla)\mathbf{u}$ in the sense of the distribution.

Considerations above yields the following conclusion when one passes to the limit in (5.16): $\forall\,\varphi$ periodic, with C^∞ class in space and time and with compact support in time,

$$< \partial_t\omega,\varphi > + < (\mathbf{u}\nabla)\omega,\varphi > +\nu\int_0^\infty\int_Q \nabla\omega:\nabla\varphi = \int_0^\infty\int_Q (\omega\nabla)\mathbf{u}.\varphi. \tag{5.48}$$

To finish the proof of Theorem 5.1, one has to deal with the initial data. Notice that we cannot make sure that the sequence $(A_{\delta_n}^{-1}\omega_0)_{n\in I\!N}$ converges strongly in L^1 to ω. We do not need this information. Indeed, let φ a C^∞ in space-time vector field, but not with time compact support and such that $\varphi(T,\cdot)=0$ for some $T>0$. One has

$$< \partial_t\omega_n,\varphi >= \int_Q A_{\delta_n}^{-1}\omega_0\,.\varphi - \int_0^T\int_Q \omega_n.\partial_t\varphi.$$

Of course,

$$\lim_{n\to\infty}\int_0^T\int_Q \omega_n.\,\partial_t\varphi = \int_0^T\int_Q \omega.\,\partial_t\varphi.$$

Remark now that

$$\int_Q A_{\delta_n}^{-1}\omega_0\,.\,\varphi = -\int_Q A_{\delta_n}^{-1}\mathbf{u}_0\,.\nabla\times\varphi.$$

Thanks to Lemma 3.1 and $\mathbf{u}_0\in L^2_x$, $(A_{\delta_n}^{-1}\mathbf{u}_0)_{n\in I\!N}$ converges strongly to $\mathbf{u}_0$ in L^2_x. Therefore,

$$\lim_{n\to\infty}\int_Q A_{\delta_n}^{-1}\mathbf{u}_0\,.\nabla\times\varphi = \int_Q \mathbf{u}_0\,.\nabla\times\varphi = -\int_Q \omega_0.\varphi,$$

yielding

$$\lim_{n\to\infty} < \partial_t\omega_n,\varphi >= \int_Q \omega_0.\varphi - \int_0^T\int_Q \omega.\,\partial_t\varphi.$$

Now we are totaly sure that ω is a distributional solution to (5.5) realized as limit of the approximations (5.16). The proof of Theorem 5.1 is now complete.

5.5 First conclusion : end of the proof of Theorem 1.1

The proof of Theorem 1.1 announced in the introduction is almost finished. It remains to check (1.14), (1.15) and (1.16).

The fact that $p\in L_t^{5/4}(W_x^{1,5/4})$, that is (1.15), is a direct consequence of (4.3) with $r=10/3$.

We note now that

$$\nabla\times\nabla\times\mathbf{v} = \nabla(\nabla\cdot\mathbf{v}) - \Delta\mathbf{v}. \tag{5.49}$$

Thus, combined with the incompressible constrain, (5.1) yields

$$-\Delta\mathbf{u} = \nabla\times\omega. \tag{5.50}$$

Therefore, (1.14) holds, that is

$$\mathbf{u} \in \bigcap_{p<5/4} L^p_t(W^{2,p}_x).$$

This is a direct consequence of (5.10) (see also Remark 4.2) . Now (1.16), that is $\partial_t u \in \bigcap_{p<5/4} L^p_{t,x}$, is obvious.

Remark 5.4 *Thanks to the hypothesis "$\omega_0 \in L^1_x$", we have gain 2 space-derivative for the velocity but we did not have gain anything for the pressure, we mean better than* (4.3)*. This is due to the fact that we did not have gain enough regularity in time.*

6 Further remarks on uniqueness

6.1 Orientations

Let $\mathbf{u}_0 \in H$. Assume that any $\mathbf{u} \in \mathcal{V}(\mathbf{u}_0)$ satisfies the following two hypothesis.

(H1) $\mathbf{u} \in L^\infty_t(L^3_x)$*, and there exists a LES sequence* $(\mathbf{w}_n)_{n\in\mathbb{N}}$ *corresponding to* $\mathbf{u}$ *which is bounded in* $L^\infty_t(L^3_x)$.

(H2) *For any* $\mathbf{u} \in \mathcal{V}(\mathbf{u}_0)$*, one has*

$$\lim_{k\to\infty} ||\mathbf{u} - T_k(\mathbf{u})||_{L^\infty_t(L^3_x)} = 0. \tag{6.1}$$

We shall prove the following statements, that we have summerized in Proposition 1.1.

Proposition 6.1 *Let* $\mathbf{u}_0 \in V$ *such that* **(H1)** *holds. Let* $\mathbf{u} \in \mathcal{V}(\mathbf{u}_0)$*. Then*

$$\frac{D\mathbf{u}}{Dt} = \partial_t \mathbf{u} + (\mathbf{u}\nabla)\mathbf{u} \in L^2_t(W'), \tag{6.2}$$

where

$$W = \{\mathbf{v} \in H^1_x;\ \nabla\cdot\mathbf{v} = 0\}. \tag{6.3}$$

The energy equality is satisfied $\forall\, t > 0$,

$$\frac{1}{2}\int_Q |\mathbf{u}(t,x)|^2 dx + \nu \int_0^t \int_Q |\nabla \mathbf{u}(t',x)|^2 dx dt' = \frac{1}{2}\int_Q |\mathbf{u}_0(x)|^2 dx, \tag{6.4}$$

and the sequence of LES velocities $(\mathbf{w}_n)_{n\in\mathbb{N}}$ *converges strongly towards* $\mathbf{u}$ *in* $L^2([0,T], H^1_x)$ *for each* $T > 0$.

Remark 6.1 *Notice that we do not need* **(H2)** *to prove* (6.2) *and* (6.4) *and the strong convergence of the approximations, for which the* $L^\infty_t(L^3_x)$ *environment is enough.*

Proposition 6.2 *Let* $\mathbf{u}_0 \in V$ *such that* **(H1)** *and* **(H2)** *hold. Then card(*$\mathcal{V}(\mathbf{u}_0)$*)* $=$ *card(*$\mathcal{P}(\mathbf{u}_0)$*)* $=$ *1, that means that there is exactly one LES solution to* (1.1).

Remark 6.2 *The results of Proposition 6.1 and Proposition 6.2 are obtain without the assumption* $\omega_0 \in L^1_x$.

6.2 Regularity of the acceleration and convergence of the energies

We start with the following classical elementary result;

Lemma 6.1 *For each function v one has*

(6.5)
$$||v||^2_{L^4_x} \leq ||v||_{L^3_x}||v||_{L^6_x} \leq C||v||_{L^3_x}||v||_{H^1_x}.$$

Proof. For the simplicity, assume $v \geq 0$. By Hölder inequality,

$$\int_Q v^4 = \int_Q v^2.v^2 \leq \left(\int_Q v^3\right)^{2/3}\left(\int_Q v^6\right)^{1/3},$$

and (6.5) follows, by using the Sobolev inequality.

Lemma 6.2 *Let $\mathbf{u}_0 \in V$ such that* **(H1)** *holds. Let $\mathbf{u} \in \mathcal{V}(\mathbf{u}_0)$ and $(\mathbf{w}_n)_{n\in I\!N}$ the corresponding LES sequence. Then the sequence $\left(\partial_t \mathbf{w}_n + \nabla \cdot (A^{-1}_{\delta_n}(\mathbf{w}_n\mathbf{w}_n))\right)_{n\in I\!N}$ is bounded in $L^2_t(W')$ and converges weakly in $L^2(W')$ to $\dfrac{D\mathbf{u}}{Dt} = \partial_t\mathbf{u} + (\mathbf{u}\nabla)\mathbf{u} \in L^2_t(W')$.*

Proof. Thanks to its regularity (5.17) and (5.18), each $\mathbf{w}_n$ satisfies

(6.6)
$$\forall\, \mathbf{v} \in W, \quad <\partial_t\mathbf{w}_n, \mathbf{v}> = < C_n(\mathbf{w}_n), \mathbf{v}> - < D_n(\mathbf{w}_n), \mathbf{v}>,$$

where

(6.7)
$$< C_n(\mathbf{w}_n), \mathbf{v}> = \int_0^\infty \int_Q A^{-1}_{\delta_n}(\mathbf{w}_n\mathbf{w}_n) : \nabla\mathbf{v},$$

(6.8)
$$< D_n(\mathbf{w}_n), \mathbf{v}> = \nu\int_0^\infty \int_Q \nabla\mathbf{w}_n : \nabla\mathbf{v}.$$

Assumption **(H1)** combined with (2.6), (3.9), (3.10) and (6.5) make sure that the sequence $(\mathbf{w}_n)_{n\in I\!N}$ is bounded in $L^4_{t,x}$. Hence, thanks to (3.17), $(A^{-1}_{\delta_n}(\mathbf{w}_n\mathbf{w}_n))_{n\in I\!N}$ is bounded in $L^2_{t,x}$. It is easy checked that this sequence $(A^{-1}_{\delta_n}(\mathbf{w}_n\mathbf{w}_n))_{n\in I\!N}$ converges weakly in $L^2_{t,x}$ to $\mathbf{uu}$. Thus, $(C_n(\mathbf{w}_n))_{n\in I\!N}$ converges weakly in $L^2(W')$ to $C(\mathbf{u})$ where

(6.9)
$$\forall\, \mathbf{v} \in W, \quad \lim_{n\to\infty} < C_n(\mathbf{w}_n), \mathbf{v}> = < C(\mathbf{u}), \mathbf{v}> = \int_0^\infty \int_Q \mathbf{uu} : \nabla\mathbf{v}.$$

We already know that

(6.10)
$$\forall\, \mathbf{v} \in W, \quad \lim_{n\to\infty} < D_n(\mathbf{w}_n), \mathbf{v}> = \int_0^\infty \int_Q \nabla\mathbf{u} : \nabla\mathbf{v} = < D(\mathbf{u}), \mathbf{v}> .$$

The sequence $\left(\partial_t \mathbf{w}_n + \nabla \cdot (A^{-1}_{\delta_n}(\mathbf{w}_n\mathbf{w}_n))\right)_{n\in I\!N}$ is now clearly bounded in $L^2_t(W')$. Then, passing to the limit in (6.6) thanks to the already known fact that $(\partial_t\mathbf{w}_n)_{n\in I\!N}$ converges to $\partial_t\mathbf{u}$ in $\mathcal{D}'$ yields

(6.11)
$$\forall\, \mathbf{v} \in W, \quad <\partial_t\mathbf{u}, \mathbf{v}> = < C(\mathbf{u}), \mathbf{v}> - < D(\mathbf{u}), \mathbf{v}> .$$

Thus in the sense of $L^2(W')$,

$$\frac{D\mathbf{u}}{Dt} = \partial_t\mathbf{u} + C(\mathbf{u}) \in L^2(W'),$$

and the convergence of $\left(\partial_t \mathbf{w}_n + \nabla\cdot(A_{\delta_n}^{-1}(\mathbf{w}_n\mathbf{w}_n))\right)_{n\in\mathbb{N}}$ towards $\dfrac{D\mathbf{u}}{Dt}$ in $L^2(W')$ weak is the consequence of (6.11), (6.6), (6.9) and (6.10) and the lemma is proved.

Proof of Proposition 6.1. Let $t>0$. One has $\dfrac{D\mathbf{u}}{Dt}\in L^2([0,t],W')$ and $u\in L^2([0,t],H^1_x)\cap L^\infty([0,t],L^2_x)$. Then, because $\mathbf{u}_0\in V$,

$$\left\langle \frac{D\mathbf{u}}{Dt},\mathbf{u}\right\rangle = \frac{1}{2}\int_Q |\mathbf{u}(t,x)|^2\,dx - \frac{1}{2}\int_Q |\mathbf{u}_0(x)|^2\,dx, \tag{6.12}$$

see for instance in the book of J. -L. Lions [18]. Now, (6.4) is clear by taking the "*authorized*" $\mathbf{u}$ as test function in (6.11).

It remains to prove the strong convergence of the sequence $(\mathbf{w}_n)_{n\in\mathbb{N}}$ to $\mathbf{u}$ in $L^2([0,T],H^1_x)$ for $T>0$ fixed. Let $T_0>T$. When one combines (3.10) with (6.4) one sees that $\mu_1=0$ and $\mu_2=0$, with the notations of (3.14). Hence, thanks to (3.12), one has

$$\lim_{n\to\infty}\int_0^{T_0}\int_0^t\int_Q |\nabla\mathbf{w}_n(t',x)|^2\,dx\,dt'dt = \int_0^{T_0}\int_0^t\int_Q |\nabla\mathbf{u}(t',x)|^2\,dx\,dt'dt. \tag{6.13}$$

It is a classical exercice to check that (6.13) yields the strong convergence of $(\mathbf{w}_n)_{n\in\mathbb{N}}$ to $\mathbf{u}$ in $L^2([0,T],H^1_x)$ for each $T<T_0$. The proof of Proposition 6.1 is now complete because the result does not depend on the choice of T_0.

6.3 Uniqueness result

We prove in this subsection Proposition 6.2. Notice that thanks to (3.32) and (4.2), it is enough to prove that card$\mathcal{V}(\mathbf{u}_0)=1$. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ both in $\mathcal{V}(\mathbf{u}_0)$, p_1 and p_2 the corresponding pressures. Note $\delta\mathbf{u}=\mathbf{u}_2-\mathbf{u}_1$, $\delta p = p_2-p_1$. Then one has

$$\partial_t\delta\mathbf{u} + (\mathbf{u}_2\nabla)\delta\mathbf{u} - \nu\Delta\delta\mathbf{u} + \nabla\delta p = -(\delta\mathbf{u}\nabla)\mathbf{u}_1. \tag{6.14}$$

Thanks to (6.11) and Proposition 6.1, $\delta\mathbf{u}$ is an *authorized* test function in (6.14). Using it yields

$$\frac{d}{2dt}\int_Q |\delta\mathbf{u}|^2 + \nu\int_Q |\nabla\delta\mathbf{u}|^2 = \int_Q (\delta\mathbf{u})\mathbf{u}_1 : \nabla\delta\mathbf{u}. \tag{6.15}$$

Then one has

$$\left\{\begin{array}{l} \dfrac{d}{2dt}\displaystyle\int_Q |\delta\mathbf{u}|^2 + \nu\int_Q |\nabla\delta\mathbf{u}|^2 \le \\ \displaystyle\int_Q |(\delta\mathbf{u})(T_k\mathbf{u}_1) : \nabla\delta\mathbf{u}| + \int_Q |(\delta\mathbf{u})(\mathbf{u}_1 - T_k\mathbf{u}_1) : \nabla\delta\mathbf{u}|. \end{array}\right. \tag{6.16}$$

Thus, by using Hölder inequality,

$$\left\{\begin{array}{l} \dfrac{d}{2dt}\displaystyle\int_Q |\delta\mathbf{u}|^2 + \nu\int_Q |\nabla\delta\mathbf{u}|^2 \le \\ k\displaystyle\int_Q |\delta\mathbf{u} : \nabla\delta\mathbf{u}| + ||\delta\mathbf{u}||_{L^6_x}||\mathbf{u}_1 - T_k\mathbf{u}_1||_{L^3_x}||\nabla\delta\mathbf{u}||_{L^2_x}. \end{array}\right. \tag{6.17}$$

Then by Sobolev inequality, using hypothesis **(H2)** and (6.1) and chossing k such that $||\mathbf{u}_1 - T_k\mathbf{u}_1||_{L^\infty(L^3_x)} \le \nu/2C$ (C = Sobolev's constant), one deduces from (6.17)

$$\frac{d}{2dt}\int_Q |\delta\mathbf{u}|^2 + \frac{\nu}{2}\int_Q |\nabla\delta\mathbf{u}|^2 \le k\int_Q |\delta\mathbf{u} : \nabla\delta\mathbf{u}|. \tag{6.18}$$

Finally, Young by Young inequality one obtains

$$\frac{d}{2dt}\int_Q |\delta\mathbf{u}|^2 + \frac{\nu}{4}\int_Q |\nabla\delta\mathbf{u}|^2 \leq \frac{4k^2}{\nu}\int_Q |\delta\mathbf{u}|^2 \tag{6.19}$$

One concluded that $\delta\mathbf{u} = 0$ thanks to Gronwall's lemma and the fact that $\delta\mathbf{u}_{t=0} = 0$ and Proposition 6.2 is proved.

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